PHYSICAL REVIEW E, VOLUME 64, 055601(R)

Energy of internal modes of nonlinear waves and complex frequencies due to symmetry breaking

Dmitry V. Skryabin[1,2]
[1]*Department of Physics, University of Bath, Bath BA2 7AY, United Kingdom*
[2]*Department of Physics and Applied Physics, University of Strathclyde, Glasgow G4 0NG, United Kingdom*
(Received 14 May 2001; published 16 October 2001)

Considering a class of Hamiltonian systems, it is demonstrated that energy of the internal modes with real frequencies supported by nonlinear waves and appearing due to perturbations breaking a continuous symmetry has its sign determined by the symmetry itself, independently of the nature of the perturbations. In particular, it is shown that negative energy modes emerge as a result of the breaking of the phase symmetry in the perturbed nonlinear-Schrödinger equation. An expression for energy of the Vakhitov-Kolokolov internal modes is also derived. Comparative analysis of the energy signs of the internal modes in these two cases explains the ubiquity of instabilities with complex frequencies of solitary and continuous waves in systems with broken phase symmetry.

DOI: 10.1103/PhysRevE.64.055601 PACS number(s): 05.45.Yv, 11.30.Qc, 42.65.Tg

Studies of the internal modes supported by nonlinear waves of different physical origin have attracted a great deal of attention over recent decades. The presence of these modes is responsible, e.g., for persistent oscillatory dynamics [1,2] and is directly linked to a most important problem of nonlinear dynamics—stability of an equilibrium [2–5].

It is clear that internal modes being excited by imperfections in initial conditions should carry certain energy. The amount of energy carried by a particular internal mode depends on the strength and geometry of the initial perturbations and cannot be considered as an intrinsic characteristic of the system and its equilibria. On the contrary, the sign of the energy of an excited mode does not generally depend on the magnitude of the perturbation. Therefore, analysis of these signs, which are often referred in mathematical literature as signatures or Krein-signs, can be expected to highlight important qualitative information, for reviews see [4,5].

The notion of energy of internal modes is similar to the quantum-mechanical definition of energy through the eigenvalues of the Hamiltonian and it can be considered as a generalization of the latter to the class of non-self-adjoint (non-Hermitian), but still Hamiltonian problems. One of the striking predictions of the energy analysis is that frequency resonances of the internal modes with *opposite energy signs* (opposite signatures), is the most typical scenario leading to the appearance of complex frequencies in the spectrum of an equilibrium [4,5]. This type of instability can be interpreted as mutual annihilation of a particle with its antiparticle. Analysis of the energy of the internal modes has been used, e.g., to interpret instabilities of plasmas [4,6] and water waves [4,7], vortices in superfluids [8,9], and optical solitons [10] and breathers [2].

One of the key problems, which has remained unaddressed so far, is whether there are any universal mechanisms governing the appearance of the modes with opposite energy signs leading to their subsequent resonances. In this work we will demonstrate that one of such mechanisms is the breaking of the continuous symmetry of the governing equations. Our analysis will be fairly general and supported by consideration of such ubiquitous examples as the perturbed nonlinear Schrödinger (NLS) equation and degenerate three wave mixing, which are applicable to many problems in optics, fluid dynamics, plasma physics, and other fields.

We start by considering a Hamiltonian system in the form

$$i\partial_t\vec{\psi}+\hat{\eta}\frac{\delta H}{\delta\vec{\psi}^*}=0, \tag{1}$$

describing the evolution of N interacting waves with complex amplitudes ψ_j, $\vec{\psi}=(\psi_1\dots\psi_N,\psi_1^*\dots\psi_N^*)^T$, $\hat{\eta}=\mathrm{diag}(-\hat{I},\hat{I})$ is the structure matrix, $\hat{I}$ is the $N\times N$ identity operator, H is the energy functional (or Hamiltonian), and $\delta H/\delta\vec{\psi}^*=(\delta H/\delta\psi_1^*,\dots\delta H/\delta\psi_N^*,\delta H/\delta\psi_1,\dots\delta H/\delta\psi_N)^T$. Below we often use the identity $\hat{\eta}\hat{\eta}=\mathrm{diag}(\hat{I},\hat{I})$.

To study the stability of an equilibrium state $\vec{\psi}_e$ found from the condition $\delta H/\delta\vec{\psi}^*=0$, we make the substitution $\vec{\psi}=\vec{\psi}_e+\vec{a}$ into H and disregard all terms of order higher than second in the components of $\vec{a}$. This procedure leaves us with linear equations describing evolution of the perturbation vector $\vec{a}$:

$$i\partial_t\vec{a}=-\hat{\eta}\frac{\delta B}{\delta\vec{a}^*}=-\hat{\eta}\hat{B}\vec{a}, \tag{2}$$

where $B=\frac{1}{2}\langle\vec{a}|\hat{B}\vec{a}\rangle$ is the quadratic part of the Hamiltonian, $\hat{B}$ is the linear $2N\times 2N$ self-adjoint operator

$$\hat{B}=\begin{bmatrix}\hat{S} & \hat{R}\\ \hat{R}^* & \hat{S}^*\end{bmatrix},\quad \hat{s}_{ij}=\frac{\delta^2 H}{\delta\psi_i^*\,\delta\psi_j},\quad \hat{r}_{ij}=\frac{\delta^2 H}{\delta\psi_i^*\,\delta\psi_j^*}, \tag{3}$$

and $\hat{S}$, $\hat{R}$ are $N\times N$ matrix operators with elements $\hat{s}_{ij}$, $\hat{r}_{ij}$. $\langle\dots|\dots\rangle$ defines scalar product, $\langle\vec{f}|\vec{g}\rangle=\int dV\Sigma_i f_i^* g_i$. Second variations in Eqs. (3) are taken for $\vec{\psi}=\vec{\psi}_e$. Assuming $\vec{a}=\vec{\xi}\exp(i\omega t)+\hat{\tau}\vec{\xi}^*\exp(-i\omega^* t)$, where $\hat{\tau}$ is the transposition matrix, $\hat{\tau}=\mathrm{antidiag}(\hat{I},\hat{I})$, we derive the non-self-adjoint eigenvalue problem

$$\hat{\eta}\hat{B}\vec{\xi}_n=\omega_n\vec{\xi}_n, \tag{4}$$

where $\vec{\xi}_n$ are eigen- or normal modes of the perturbations and ω_n are their eigenfrequencies. Internal modes are eigenmodes with real frequencies belonging to the discrete spectrum.

Perturbing an equilibrium state $\vec{\psi}_e$, one generally either adds or subtracts energy. Assuming that $\vec{a}(t=0)=\vec{\xi}_n+\hat{\tau}\vec{\xi}_n^*$, we can define the energy ϵ_n of the normal perturbation as

$$\epsilon_n=\langle\vec{\xi}_n|\hat{B}\vec{\xi}_n\rangle=\omega_n\langle\vec{\xi}_n|\hat{\eta}\vec{\xi}_n\rangle. \tag{5}$$

ϵ_n is simply the difference between the Hamiltonians for the perturbed and unperturbed equilibria.

Significant insight into Eq. (5) can be achieved, using the biorthogonality of the eigenmodes of $\hat{\eta}\hat{B}$ and of the adjoint operator $\hat{B}\hat{\eta}$, i.e., $\langle\vec{\xi}_n|\vec{v}_{n'}\rangle=0$, where $n\neq n'$ and $\vec{v}_n$ obeys $(\hat{\eta}\hat{B})^\dagger\vec{v}_n=\hat{B}\hat{\eta}\vec{v}_n=\omega_n^*\vec{v}_n$. If ω_n is real, then $\vec{v}_n=\hat{\eta}\vec{\xi}_n$ and if ω_n is complex or imaginary, then $\vec{v}_n=\hat{\eta}\vec{\xi}_n^*$. In the latter case, $\hat{\eta}\vec{\xi}_n$ is also an eigenmode of $\hat{B}\hat{\eta}$, but with eigenfrequency ω_n. Hence if $Im\,\omega_n\neq 0$, then biorthogonality implies $\langle\vec{\xi}_n|\hat{\eta}\vec{\xi}_n\rangle=\epsilon_n=0$, i.e., *modes with complex or imaginary frequencies carry zero energy*, see also [5,8]. For modes with real frequencies, the magnitude, but not the sign of ϵ_n, can be scaled. However, it is more convenient for us to continue without any normalization, keeping in mind that sgn(ϵ_n), not $|\epsilon_n|$, has paramount importance. For more details of the interplay between the biorthogonality and energy signs, see [8].

We assume now that the equilibrium under consideration is spectrally stable, i.e., it has real spectrum, and that H can be present in the form $H=H_0+\mu H_{sb}$. H_0 is invariant with respect to a continuous one-parameter symmetry group $\mathcal{G}_\phi$, and H_{sb} describes corrections breaking this symmetry. Note that previous papers exploring the energy sign ideas in the context of the stability of solitary waves, see, e.g., [2,10] and references therein, have not considered the problem of the energy signs of the internal modes emerging due to perturbations breaking the continuous symmetry group of the Hamiltonian. The parameter μ characterizes the strength of the symmetry breaking perturbations and it is assumed to be a small parameter in our analysis. $\hat{B}$ also can be presented in the form $\hat{B}_0+\mu\hat{B}_{sb}$, where $\hat{B}_{sb}$ contains the corrections to the Hamiltonian and the equilibrium. It is an important fact that the energy sign of the internal modes preserves as parameters vary as long as there is no frequency resonance with another mode, in which case energy can become zero [5]. Thus if the energy sign is found for $\mu\ll 1$, it will remain the same even outside the region of validity of the asymptotic expansion.

We also assume that the infinitesimal transformation $\mathcal{G}_{\delta\phi}$ applied to the equilibrium $\vec{\psi}_e$ generates a Goldstone or neutral mode $\vec{U}_0$ such that $\hat{B}_0\vec{U}_0=0$. It is obvious that $\langle\vec{U}_0|\hat{\eta}\vec{U}_0\rangle\equiv 0$. Therefore, the generalized eigenvector $\vec{U}_1$, defined as the solution of $\hat{\eta}\hat{B}\vec{U}_1=\vec{U}_0$, exists so that the zero eigenfrequency of $\hat{\eta}\hat{B}$ with eigenvector $\vec{U}_0$ has algebraic multiplicity 2.

We aim to calculate the frequencies and energy signs of the eigenmodes of $\hat{\eta}\hat{B}$, replacing $\vec{U}_0$ under the action of the symmetry breaking perturbations. In order to do this we expand the solution of Eq. (4) as an asymptotic series in μ assuming that $\omega\sim\sqrt{\mu}$. Keeping the first two terms we find that

$$\vec{\xi}=\vec{U}_0+\omega\vec{U}_1+\ \ldots. \tag{6}$$

Next, we substitute Eq. (6) into the orthogonality condition

$$\mu\langle\hat{\eta}\hat{B}_{sb}\vec{\xi}|\hat{\eta}\vec{U}_0\rangle=\omega\langle\vec{\xi}|\hat{\eta}\vec{U}_0\rangle, \tag{7}$$

which is derived by multiplying Eq. (4) by $\hat{\eta}\vec{U}_0$. To leading order we find that frequencies of the modes appearing due to symmetry breaking can be found from

$$\omega_{sb}^2\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle=\mu\langle\hat{B}_{sb}\vec{U}_0|\vec{U}_0\rangle+O(\mu^{3/2}) \tag{8}$$

and the corresponding energy is

$$\epsilon_{sb}=\omega_{sb}(\omega_{sb}+\omega_{sb}^*)\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle+O(|\omega|^3). \tag{9}$$

All particular features of the perturbations are hidden inside ω_{sb} and their role is only in determining whether ω is real or imaginary. If $Im\,\omega_{sb}=0$, then $\epsilon_{sb}\sim\omega_{sb}^2$ and therefore *the energy sign of the emerging spectrally stable internal modes is independent from the symmetry-breaking perturbations $\hat{\eta}\hat{B}_{sb}$, but is solely determined by the properties of the broken symmetry reflected in the structure of the operator $\hat{\eta}\hat{B}_0$ and its eigenmodes*. Thus,

$$\mathrm{sgn}(\epsilon_{sb})=\mathrm{sgn}\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle. \tag{10}$$

Note that the latter result for the energy sign is valid for arbitrary large μ providing that no frequency resonances with other internal modes take place. If $Re\,\omega_{sb}=0$ and $Im\,\omega_{sb}\neq 0$, i.e., symmetry breaking leads to spectral instability, then $\epsilon_{sb}=0$.

To illustrate applications and further develop the above ideas, we consider the fundamental examples of the two-dimensional (2D) NLS equation and three wave mixing. 2D NLS with generalized nonlinearity, can be derived from the Hamiltonian

$$H_0=\int\int dxdy\left\{|\vec{\nabla}\psi|^2+\theta|\psi|^2-\int_0^{|\psi|^2}dIf(I)\right\}. \tag{11}$$

We assume below that $0<\theta<\max_I f(I)$ and $\partial_I f>0$, which ensures existence of bright solitary solutions [3,11]. The symmetry of interest is

$$\psi\rightarrow\psi e^{i\phi}, \tag{12}$$

with the corresponding Goldstone mode $\vec{U}_0=(\psi_0,-\psi_0)^T$. Here ψ_0 is a real function obeying $\vec{\nabla}^2\psi_0=(\theta-f(\psi_0^2))\psi_0$, which characterizes the transverse profile of the soliton. The generalized eigenvector associated with $\vec{U}_0$ is $\vec{U}_1 = \partial_\theta(\psi_0, \psi_0)^T$.

Typical perturbations breaking symmetry (12) and preserving the Hamiltonian structure of the equations are either external,

$$H_{sb}^{(e)} = -\int\int dxdy\{\psi^* + \psi\} \tag{13}$$

or parametric,

$$H_{sb}^{(p)} = -\frac{1}{2}\int\int dxdy\{\psi^*\psi^* + \psi\psi\} \tag{14}$$

forcing. In the context of nonlinear optics, Hamiltonians $H_0 + \mu H_{sb}^{(e,p)}$ describe, respectively, nonlinear optical cavities with intensity dependent purely dispersive nonlinearity excited by an external monochromatic pump [12] and intracavity parametric wave mixing when the pump field is far off-resonance or not resonated at all [13]. The parameter θ has physical meaning as the frequency detuning of the field from the cavity resonance. The symmetry breaking perturbations (13) and (14) also arise in the plasma physics [14], dynamics of fluids [15], spin waves in ferromagnets and other contexts [16].

It follows from Eq. (10) that the pair of the zero-energy modes $\vec{U}_0$ transforms into the internal modes carrying energy with signs given by

$$\mathrm{sgn}(\epsilon_{sb}) = \mathrm{sgn}(-\partial_\theta Q), \tag{15}$$

where $Q = \int\int dxdy\,\psi_0^2$ is the power or number of particles invariant. Thus if the Vakhitov-Kolokolov (VK) stability criterion [11]

$$-\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle = \partial_\theta Q > 0 \tag{16}$$

is satisfied for $\mu = 0$, then the breaking of the phase symmetry leads to the emergence of *negative-energy modes*. Equations (9), (10) and (16) represent main conceptually different results of this work. The rest of the paper is largely devoted to the illustration of their applicability.

It is clear that Eq. (8) loses its validity when $\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle$ either changes sign or is simply close to zero. Therefore, the perturbation expansion capturing symmetry breaking effects has to be modified in the vicinity of such a point. However, before proceeding with these modifications, it is instructive to consider the situation, when $\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle$ is small, but no symmetry breaking is present.

The system $\hat{\eta}\hat{B}\vec{U}_2 = \vec{U}_1$ is solvable if $\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle = 0$. Then equation $\hat{\eta}\hat{B}\vec{U}_3 = \vec{U}_2$ is also solvable without any additional constraint, simply because $\langle\vec{U}_2|\hat{\eta}\vec{U}_0\rangle \equiv 0$. Therefore the zero eigenfrequency with eigenvector $\vec{U}_0$ has now algebraic multiplicity 4. If we assume $\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle \sim \omega^2$, then the solution of Eq. (4) with $\hat{B} = \hat{B}_0$, up to the fourth order, is

$$\vec{\xi} = \vec{U}_0 + \omega\vec{U}_1 + \omega^2\vec{U}_2 + \omega^3\vec{U}_3 + O(|\omega|^4). \tag{17}$$

Substituting Eq. (17) into the right-hand side of the orthogonality condition (7), we find that $\omega_{vk}^2\langle\vec{U}_2|\hat{\eta}\vec{U}_1\rangle + \langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle = O(|\omega_{vk}|^4)$. Thus, when $\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle$ passes through zero, then two purely real eigenvalues pass through zero and move along the imaginary axis. The above consideration is just another way to prove validity of the VK criterion (16).

Using expansion (17), we can calculate energy ϵ_{vk} carried by the internal modes causing the VK instability. Substituting Eq. (17) into Eq. (5), we find

$$\epsilon_{vk} = 2\omega_{vk}(\omega_{vk}^3 + \omega_{vk}^{*3})\langle\vec{U}_2|\hat{\eta}\vec{U}_1\rangle + O(|\omega_{vk}|^5). \tag{18}$$

The criterion (16) implies practically the spectral stability of the solitary waves in many problems, despite the fact that its sufficiency can be rigorously proved only in a limited number of cases [17]. Stability necessary requires $\omega_{vk}^2 > 0$, which together with (16), readily implies

$$\mathrm{sgn}(\epsilon_{vk}) = \mathrm{sgn}\langle\vec{U}_2|\hat{\eta}\vec{U}_1\rangle > 0. \tag{19}$$

In fact, positivity of the energy carried by the VK modes also readily follows from the variational methods of the proof of the VK criterion given in [17].

It follows from the preceding that if perturbations breaking the phase symmetry have been added into a system with a VK threshold, then coexistence of the modes with positive and negative energies and subsequent complex-frequency instability should be expected. A famous example of a system with four-fold degeneracy of $\vec{U}_0$ is provided by the ground-state solitary wave of the Hamiltonian (11) with $f(I) = I$. Any perturbation consisting of saturating nonlinearity, the simplest of which is $f(I) - I = -\beta I^2$, $0 < \beta \ll 1$, replaces the critical four-fold degeneracy by a noncritical double degeneracy and leads to emergence of the low-frequency positive-energy VK modes. An interesting property of 2D NLS with $f = I$ is that the second generalized eigenvector can be found explicitly [3]: $\vec{U}_2 = 1/8\theta r^2\vec{U}_0$.

To verify that symmetry breaking close to the VK threshold $\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle \sim \omega^2$ does indeed lead to a complex-frequency instability in driven 2D NLS, we use again perturbations (13), (14), and expansion (17). Assuming that $\mu^{1/4} \sim \omega$, we find that the orthogonality condition (7) gives

$$\langle\hat{B}_{sb}^{(e,p)}\vec{U}_0|\vec{U}_0\rangle = \omega^4\langle\vec{U}_2|\hat{\eta}\vec{U}_1\rangle + \omega^2\langle\vec{U}_1|\hat{\eta}\vec{U}_0\rangle. \tag{20}$$

Then, taking $\beta \sim \sqrt{\mu}$, we show that the solitary solutions themselves are $\psi_\pm^{(e,p)} = e^{i\phi_\pm^{(e,p)}}\{\sqrt{\theta}\Psi_0 + \beta\theta^{3/2}\Psi_1 \mp (\mu/\theta)\Psi_2^{(e,p)} + \beta^2\Psi_3\} + O(\mu^{3/2})$, where $\Psi_2^{(p)} = \frac{1}{2}\sqrt{\theta}[\Psi_0 + \rho(d\Psi_0/d\rho)]$ and $\rho = r\sqrt{\theta}$. $\phi_\pm^{e,p}$ are the phases locked by the perturbations $\phi_+^{(e,p)} = 0$, $\phi_-^{(e)} = \pi$, and $\phi_-^{(p)} = \pi/2$, i.e., for each of the cases the phase can be locked to two physically distinct values. Function Ψ_3 is irrelevant for our purposes and $\Psi_{0,1}$, $\Psi_2^{(e)}$ are solutions of the parameter independent equations $[\hat{D} + \Psi_0^2]\Psi_0 = 0$, $[\hat{D} + 3\Psi_0^2]\Psi_1 = \Psi_0^5$, $[\hat{D} + 3\Psi_0^2]\Psi_2^{(e)} = 1$ with $\hat{D} \equiv \partial_\rho^2 + (1/\rho)\partial_\rho - 1$. Numerically solving the latter, we find that $\langle\hat{B}_{sb,\pm}^{(p)}\vec{U}_0|\vec{U}_0\rangle \simeq \pm 29.8\pi\mu$, $\langle\hat{B}_{sb,\pm}^{(e)}\vec{U}_0|\vec{U}_0\rangle \simeq \pm 9.619\pi\mu/\sqrt{\theta}$, $\langle\vec{U}_2|\hat{\eta}\vec{U}_1\rangle \simeq 0.553\pi/\theta^3$, and $\langle \vec{U}_1 | \hat{\eta} \vec{U}_0 \rangle \simeq -15.084\pi\beta$. Now it can be easily shown that soliton branches with $\psi^{(e,p)} = \psi_-^{(e,p)}$ are stable when μ deviates from zero, but lose their stability due to appearance of the complex frequencies, when μ exceeds threshold values $\mu_{th}^{(p)} \simeq 3.5\theta^3\beta^2$ and $\mu_{th}^{(e)} \simeq 10.7\theta^{7/2}\beta^2$. The branches $\psi_+^{(e,p)}$ have a zero-energy unstable mode with purely imaginary frequency for any nonzero μ.

The approach developed above can be applied not only for solitary waves, but also in the simpler cases, when the equilibrium under consideration is a continuous wave. To illustrate this, one can consider the Hamiltonian describing degenerate intracavity three wave mixing [18]

$$H = \theta_\Omega |\psi_1|^2 + \theta_{2\Omega} |\psi_2|^2 + \tfrac{1}{2}(\psi_1^2 \psi_2^* + \text{c.c.}) - (\mu_1 \psi_1^* + \mu_2 \psi_2^* + \text{c.c.}). \tag{21}$$

Here $\theta_{\Omega,2\Omega}$ are detunings of the fields at frequencies Ω and 2Ω from the nearest cavity resonances, $\mu_1 = 0$ and $\mu_2 = 0$ correspond, respectively, to the case of frequency down-conversion and second-harmonic generation. For $\mu_{1,2} = 0$ the system has gauge symmetry $(\psi_1, \psi_2) \to (\psi_1 e^{i\phi}, \psi_2 e^{i2\phi})$. Introducing parameters θ and δ, $\theta_\Omega = \theta$, $\theta_{2\Omega} = 2\theta + \delta$, one can show that the VK threshold is given by $\partial_\theta(|\psi_{01}|^2 + 2|\psi_{02}|^2) = 0$, and for the solution $|\psi_{01}|^2 = 2\theta_\Omega\theta_{2\Omega}$, $|\psi_{02}| = |\theta_\Omega|$, it exists at $\theta_{\Omega,2\Omega} = 0$. Therefore VK modes are present for $\theta_{\Omega,2\Omega} \neq 0$. If any or both $\mu_{1,2} \neq 0$, then complex-frequency instability is expected and indeed happens as was found by traditional analysis [18]. If diffraction is included into Eq. (21), then the same instability of solitary structures can also be found [19].

An important ingredient, which we have disregarded above, is non-Hamiltonian perturbations, the simplest of which are linear losses. A phenomenological account of the effect of losses on the soliton spectrum can be made by replacing ω by $\omega + i\gamma$, $\gamma > 0$. This trick is often used, e.g., in textbooks on nonlinear optics, to include damping phenomena into dispersion coefficients. In soliton problems it describes a shift of the soliton spectrum by γ in the $(Re\,\omega, Im\,\omega)$ plane, which qualitatively agrees with numerical calculations [19]. An approach to the problem of non-Hamiltonian perturbations, emphasizing the role played by the energy of internal modes has been outlined in [20] and should be a guideline for future rigorous studies of the interplay between symmetry breaking and losses. Note, that previous works known to the author that explore complex-frequency instabilities due to symmetry breaking, see [19,21,22], do not contain energy based analyses, showing, however, generality of the phenomenon described above.

In summary, considering a class of Hamiltonian systems with broken symmetries, we have demonstrated that internal modes of nonlinear waves, replacing Goldstone modes, carry energy with signs that are independent of the choice of the symmetry-breaking perturbations. In particular, breaking of the phase symmetry in the NLS equation leads to appearance of the negative energy modes, which, in turn, explains the presence of complex frequency instabilities of solitary waves in a variety of physical systems reducible to the driven NLS.

The author is grateful to A.G. Vladimirov for the calculation of numerical constants and to A.R. Champneys for a careful reading of the manuscript. Support from the Royal Society of Edinburgh and UK EPSRC, Grant No. GR/N19830 is acknowledged.

[1] Y.S. Kivshar *et al.*, Phys. Rev. Lett. **80**, 5032 (1998).
[2] M. Johansson and S. Aubry, Phys. Rev. E **61**, 5864 (2000); T. Kapitula, P.G. Kevrekidis, and B.A. Malomed, *ibid.* **63**, 036604 (2001).
[3] E.A. Kuznetsov, A.M. Rubenchik, and V.E. Zakharov, Phys. Rep. **142**, 103 (1986).
[4] P.J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).
[5] R.S. MacKay, in *Stability of Equilibria of Hamiltonian Systems*, edited by R.S. MacKay and J.D. Meiss (Hilger, Bristol, 1987), p. 137.
[6] A.J. Brizard *et al.*, Phys. Rev. Lett. **77**, 1500 (1996).
[7] R.S. MacKay and P.G. Saffman, Proc. R. Soc. London, Ser. A **406**, 115 (1986).
[8] D.V. Skryabin, Phys. Rev. A **63**, 013602 (2001).
[9] D.L. Feder *et al.*, Phys. Rev. Lett. **86**, 564 (2001).
[10] Y.A. Li and K. Promislow, Physica D **124**, 137 (1998).
[11] M.G. Vakhitov and A.A. Kolokolov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **16**, 1020 (1973) [Sov. Radiophys. **16**, 783 (1973)].
[12] L.A. Lugiato and R. Lefever, Phys. Rev. Lett. **58**, 2209 (1987).
[13] S. Trillo and M. Haelterman, Opt. Lett. **23**, 1514 (1998); K. Promislow and J.N. Kutz, Nonlinearity **13**, 675 (2000).
[14] K. Nozaki and N. Bekki, Physica D **21**, 381 (1986).
[15] J.W. Miles, J. Fluid Mech. **148**, 451 (1984).
[16] D.J. Kaup and A.C. Newell, Proc. R. Soc. London, Ser. A **361**, 413 (1978).
[17] M. Grillakis, J. Shatah, and W. Strauss, J. Funct. Anal. **74**, 160 (1987); **94**, 308 (1990).
[18] K.J. McNeil, P.D. Drummond, and D.F. Walls, Opt. Commun. **27**, 292 (1978); P. Mandel and T. Erneux, Opt. Acta **29**, 7 (1982); L.A. Lugiato *et al.*, Nuovo Cimento **10**, 959 (1988).
[19] D.V. Skryabin, Phys. Rev. E **60**, R3508 (1999).
[20] R.S. MacKay, Phys. Lett. A **155**, 266 (1991).
[21] J. Guckenheimer and A. Mahalov, Phys. Rev. Lett. **68**, 2257 (1992).
[22] D.V. Skryabin, Phys. Rev. E **60**, 7511 (1999).